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The Kuratowski convergence and connected components

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ABSTRACT

We investigate the Kuratowski convergence of the connected components of the sections of a definable set applying the result obtained to semialgebraic approximation of subanalytic sets. We are led to some considerations concerning the connectedness of the limit set in general. We discuss also the behaviour of the dimension of converging sections and prove some general facts about the Kuratowski convergence in tame geometry.

1. Introduction

In [5] we dealt with the Kuratowski convergence of the sections of a subanalytic or definable (in some o-minimal structure) set providing a topological criterion for the continuity of the section and comparing it with the convergence in measure. The question of the continuity of the sections of a subanalytic set was asked by Łojasiewicz. Strangely enough it has never been thoroughly studied. In this article we investigate the behaviour of subanalytic/definable families of subanalytic or definable sets, or in other words of subanalytic or definable multifunctions (as treated in the fundamental textbook [1]). Before we explain the results enclosed herewith, we shall recall the notion of convergence we will be using.

The Kuratowski convergence of closed sets (also known as the Painlevé convergence) is closely related to the Γ-convergence of De Giorgi and optimisation which accounts for our interest for it (it is also widely used in complex analytic geometry — e.g. in Bishop's theorem, in [2] for approximation or in [7] where a Chevalley-type theorem is obtained for complex Nash sets using this convergence; actually, [10] and [3] also deals with this type of convergence). It is usually defined for a sequence of closed, nonempty sets $\{A_n\}_{n\in\mathbb{N}}$ as follows (see e.g. [9,1], each of which gives different approach and applications): $a \in \limsup A_n$ iff there exists a subsequence $\{A_{n_k}\}_k$ and a sequence of points $a_{n_k} \in A_{n_k}$ converging to a, $a \in \liminf A_n$ iff there exists a sequence of points $a_n \in A_n$ converging to a.

If $\liminf A_n = \limsup A_n = A$, the set A is called the (Kuratowski) limit of the sequence $\{A_n\}$ and denoted $\lim A_n$.

This convergence is metrisable and compact.

The Kuratowski convergence in $\mathbb{R}^m$ may also be expressed as follows: $A = \lim A_n$ iff each point $a \in A$ is the limit of a sequence of points $a_n \in A_n$, $n \in \mathbb{N}$ and for each compact set K such that $K \cap A = \varnothing$ one has $K \cap A_n = \varnothing$ for almost all indices n.

In that case we have the following easy and well-known proposition (compare [13]):

Proposition 1.1. *If $f, f_\nu : F \to \mathbb{R}^n$, $\nu \in \mathbb{N}$ are continuous functions defined on a locally closed set $F \subset \mathbb{R}^k$, then the sequence $\{f_\nu\}$ converges locally uniformly to f iff $\Gamma = \lim \Gamma_\nu$ (in $F \times \mathbb{R}^k$), where Γ and Γ_ν respectively denote the graphs of f and f_ν.*

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The Kuratowski convergence of *compact* sets is equivalent to the convergence in the Hausdorff metric ([9], §29) — for (compact) subsets of $\mathbb{R}^m$ we put

$$\mathrm{dist}_H(K,L)=\inf\bigl\{r>0 \bigm| K\subset L+\overline{\mathbb{B}}(r),\ L\subset K+\overline{\mathbb{B}}(r)\bigr\},$$

where $\overline{\mathbb{B}}(r)=\overline{\mathbb{B}}(0,r)$ denotes the closure of the unit Euclidean ball centred at zero[2] — but it need not be so if the sets are just closed, as is shown in the following example: Let A_n be the union of the semilines $(-\infty,n]\times\{0\}$ and $[n+1,+\infty)\times\{0\}$ in $\mathbb{R}^2$ with $\{(x,x-n)\mid x\in[n,n+1/2]\}\cup\{(x,-x+n+1)\mid x\in[n+1/2,n+1]\}$. Here the Kuratowski limit is equal to $A:=\mathbb{R}\times\{0\}$ while the Hausdorff distance between A and any of the sets A_n is always equal to 1.

As the epigraphs of functions studied in Γ-convergence are not compact, we prefer avoid using the Hausdorff metric in our proofs.

Instead of studying sequences of sets we will study nets (i.e. Moore–Smith sequences) of sets with indices being neighbourhoods of a given point ordered by the relation $V\prec W$ iff $V\supset W$.[3] More precisely we will be considering the following situation (as in [5]): Let $E\subset\mathbb{R}^k_t\times\mathbb{R}^m_x$ be a nonempty set, $\pi_k(t,x)=t$ the natural projection and $F:=\pi_k(E)$. For any $t\in F$ we denote by

$$E_t:=\bigl\{x\in\mathbb{R}^m \bigm| (t,x)\in E\bigr\}$$

the *section* of E at t. For simplicity, we suppose hereafter that $t_0=0\in F$. The Kuratowski convergence of the sections may now be expressed as follows:

The set E defines a net $\{E_t\}$ of sections $E_t\subset\mathbb{R}^m$ (we take a basis of neighbourhoods of $t_0=0$ in F as a directed set). We say that:

(1) $x\in\limsup E_t$ iff there is a subnet $x_{t_\alpha}\in E_{t_\alpha}$ such that $x_{t_\alpha}\to x$ when $t_\alpha\to 0$, in other words iff $\forall U\subset\mathbb{R}^m$ neighbourhood of x, $\forall V\subset\mathbb{R}^k$ neighbourhood of 0 $\exists t\in V\setminus\{0\}$: $E_t\cap U\neq\varnothing$.
(2) $x\in\liminf E_t$ iff there is a net $x_t\in E_t$ such that $x_t\to x$, in other words iff $\forall U\ni x$ $\exists V\ni 0$: $E_t\cap U\neq\varnothing$ for $t\in V\cap F\setminus\{0\}$.

Obviously we have $\liminf E_t\subset\limsup E_t$ and so E_t converges to E_0 in the sense of Kuratowski iff

$$\limsup E_t\subset E_0\subset\liminf E_t.$$

We write then $E_t\to E_0$. It is easy to see that the upper and lower Kuratowski limits are closed sets. That is why we work with E *closed*.

If E is closed, then $\lim E_t=E_0$ iff $E_0\subset\liminf E_t$ (see [5], (2.1)). However, in the category of closed definable[4]/subanalytic sets it is often more convenient to consider convergence in the sense of the upper limit.

Our first aim was to study how do the connected components behave under the operation of taking the Kuratowski limits. Suppose E is a compact definable set. The question is, does the equality $E_0=\limsup E_t$ (or at least $E_0=\lim E_t$) imply that each connected component of E_0 is the (upper) limit of a net of connected components of E_t and, besides, is the number of connected components of E_0 smaller or equal to the number of connected components of E_t for almost all t?[5] Compactness is clearly necessary when pondering such things (cf. [5]).

This is a natural question in view of the important role played by the connected components in many matters from real (subanalytical or tame) geometry. We discovered on this occasion that there is very little, if anything, written on the Kuratowski convergence in the subanalytic or definable setting, although this is perhaps the most natural type of convergence to be used for closed sets. Therefore, the next section is devoted to proving some basic properties of this convergence in the definable setting, such as the definability of the upper and lower limit (Theorem 2.5 — L. Bröcker obtained a similar result for the Hausdorff limits in the semialgebraic context, compare also J.-M. Lion and P. Speissegger [10]; Theorem 2.11 on ‘almost everywhere’, in the definable sense, continuity — this actually answers a question of L. Bröcker from [3]).

Then, in the following section we deal with the connected components of definable families of definable sets proving that the function counting their number is definable (Theorem 3.1) and answering the question asked at the beginning (Theorem 3.9, similarly as in the case of the original question of Łojasiewicz concerning the continuity, it turns out that the property we were looking for is purely topological and we do not need any special structure of the sets involved). Moreover, we complete a result of L. Bröcker on the semi-continuity of the dimension (Theorem 3.13).

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2 Usually the definition of d_H involves open balls, but the resulting metric is obviously the same and we shall need closed ones.

3 This is partly due to the fact that we will work most of the time with sets definable in some o-minimal structure and want therefore to avoid the use of sequences (for their use is unusual in this setting. Moreover, there is always a basis of definable neighbourhoods).

4 Throughout the paper by ‘definable’ we mean ‘definable in some o-minimal structure’ on $\mathbb{R}$; see [4] for definitions. In this context E may be seen as a definable family of sets $\{E_t\}_{t\in F}$. On the other hand, from the point of view of [1], E is just the graph of a (definable) multifunction $E:\mathbb{R}^k\rightrightarrows\mathbb{R}^n$. However, it seems to us that it is easier to state our results from the point of view of sections of a given set.

5 This question is related to an older topological one asking how do continua behave when passing to limits — some answers were given by L. Zoretti at the beginning of the XXth century for Hausdorff limits of compact sets, see [9], §42.II.

The next section is devoted to a simple application of Theorem 3.9, namely to semialgebraic approximation of subanalytic sets. We obtain this approximation in a rather elementary way in Theorem 4.1. Then we present another, but less 'effective', way of approximating a subanalytic set with a semialgebraic sequence.

Finally, in the last section we present some actually purely topological results we obtained as a by-product of our preceding investigations (e.g. the completion of the Zoretti Theorem in Proposition 5.1) as well as a consequence of looking for an answer to a question asked once by J.-P. Rolin. This question concerned a uniform bound on the number of connected components when we pass to the limit. An interesting, non-trivial result is presented in Theorem 5.8.

2. Preliminary results and examples

We begin with some general remarks concerning the Kuratowski convergence. Here follows a useful description of this convergence (see e.g. [12][6]).

Lemma 2.1. *Let $E \subset \mathbb{R}^k_t \times \mathbb{R}^n_x$. Then $E_0 = \lim E_t$ iff for all $x \in E_0$ there is a net $E_t \ni x_t \to x$ and for all compact $K \subset \mathbb{R}^n \setminus E_0$ there is a neighbourhood V of $0 \in \mathbb{R}^k$ such that $K \cap E_t = \varnothing$ for $t \in V$.*

Remark 2.2. Clearly, this makes sense in any locally compact, regular topological space with countable bases of neighbourhoods. Actually, many of the results presented here may be obtained, without changes in the proofs, in a general metric space setting.

One more general fact which is a kind of 'Double Limits Theorem'[7]:

Proposition 2.3. *Let $E \subset \mathbb{R}^p_\alpha \times \mathbb{R}^k_t \times \mathbb{R}^n_x$ be such that all E_α are closed. Suppose that for some α_0 there is $F_{\alpha_0} = \lim F_\alpha$, where $F_\alpha := \pi_k(E_\alpha)$ for $\pi_k(t,x) = t$. If, moreover, for any $t \in F_{\alpha_0}$ there is $(E_\alpha)_{t_\alpha} \to (E_{\alpha_0})_t$ for any net $t_\alpha \in F_\alpha$, $t_\alpha \to t$,[8] then $E_{\alpha_0} = \lim E_\alpha$.*

Before the proof we will try to make the statement clearer by explaining what it is about in the language of multifunctions: take a family of multifunctions $E_\alpha : \mathbb{R}^k \rightrightarrows \mathbb{R}^n$ ($\alpha \in \mathbb{R}^p$) with convergent domains $F_\alpha \to F_{\alpha_0}$ in $\mathbb{R}^k$. Then assuming that for any $t \in F_{\alpha_0}$ and any net $t_\alpha \in F_\alpha$, $t_\alpha \to t$, there is $E_\alpha(t_\alpha) \to E_{\alpha_0}(t)$, we conclude that the graphs of E_α converge in $\mathbb{R}^k \times \mathbb{R}^n$ to the graph of E_{α_0}.

Proof. Take a point $(t,x) \in E_{\alpha_0}$. Then, since $F_\alpha \to F_{\alpha_0}$, there exists a net $F_\alpha \ni t_\alpha \to t$ for which $(E_\alpha)_{t_\alpha} \to (E_{\alpha_0})_t$ and so there is a net $(E_\alpha)_{t_\alpha} \ni x_\alpha \to x \in (E_{\alpha_0})_t$. Therefore, $E_\alpha \ni (t_\alpha, x_\alpha) \to (t,x)$.

Let $E_{\alpha_s} \ni (t_{\alpha_s}, x_{\alpha_s}) \to (t_0, x_0)$ be a convergent subnet. Since $F_{\alpha_0} = \lim F_\alpha$ and $t_{\alpha_s} \in F_{\alpha_s}$, there must be $t_0 \in F_{\alpha_0}$. On the other hand, $\{t_{\alpha_s}\}$ is a subnet of a net $t_\alpha \in F_\alpha$ converging to t_0. Therefore, $x_0 \in \limsup_{t_\alpha \to t_0} (E_\alpha)_{t_\alpha} = (E_{\alpha_0})_{t_0}$. □

Remark 2.4. Note that $E_\alpha \to E_{\alpha_0}$ does not imply that $F_\alpha \to F_{\alpha_0}$, unless the sets are compact (the convergence is then the Hausdorff convergence and $\pi_k(E_\alpha + \mathbb{B}_{k+n}(\varepsilon)) = F_\alpha + \mathbb{B}_k(\varepsilon)$). The simplest example is to take $\alpha_0 = 0$, $E_0 = \{0\} \times \mathbb{R}$ and $E_\alpha = \{(t, t/\alpha) \mid t \in \mathbb{R}\}$ for $\alpha \neq 0$.

Note also that when $E_\alpha \to E_{\alpha_0}$, even if we assume that $F_\alpha \to F_{\alpha_0}$, the convergence $F_\alpha \ni t_\alpha \to t \in F_{\alpha_0}$ does not necessarily imply $(E_\alpha)_{t_\alpha} \to (E_{\alpha_0})_t$. To see this it suffices to take a constant family $E_\alpha = E_{\alpha_0}$ where the last set is 'discontinuous' at some $t \in F_{\alpha_0}$, see Example 3.14 with $t = 0$. The same example shows that there is no converse to the proposition (i.e. although $E_\alpha \to E_{\alpha_0}$ and $F_\alpha \to F_{\alpha_0}$, it may happen that for some $t \in F_{\alpha_0}$ the net $(E_\alpha)_{t_\alpha}$ does not converge to $(E_{\alpha_0})_t$ for any $F_\alpha \ni t_\alpha \to t$).

Moreover, the proposition is no longer true if one replaces *for any net* $F_\alpha \ni t_\alpha \to t$ by *for some net* $F_\alpha \ni t_\alpha \to t$. To show this we consider α's from the set $\{0\} \cup \bigcup \{1/\nu\}$, $E_0 = [0,1] \times \{0\}$ and $E_{1/\nu} = \{(t, t^\nu), t \in [0,1]\}$. Then $E_{1/\nu} \to E_0 \cup (\{1\} \times [0,1])$, but $F_0 = F_{1/\nu} = [0,1]$ and for any $t \in F_0$ there is a sequence $t_{1/\nu} \to t$ for which $(E_{1/\nu})_{t_{1/\nu}} = \{(t_{1/\nu}, t^\nu_{1/\nu})\} \to \{(t,0)\}$,

Now, we turn to the definable setting.

Theorem 2.5. *If $E \subset \mathbb{R}^k_t \times \mathbb{R}^n_x$ is a definable set with closed sections E_t and we denote by $F = \pi_k(E)$ its projection onto $\mathbb{R}^k$, then for each point $t_0 \in \overline{F}$ both $\limsup E_t$ and $\liminf E_t$ computed for $t \to t_0$ are definable sets.*

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[6] We thank the referee for the reference.

[7] One may consider this proposition (and the following remarks) as a result concerning the convergence of a family of fibred sets.

[8] Here $\{(E_\alpha)_{t_\alpha}\}$ is to be understood as a net indexed by the set $\{t_\alpha, \alpha\}$ directed by the basis of neighbourhoods of t. Of course, $(E_\alpha)_{t_\alpha} = E_{(\alpha, t_\alpha)}$ and F_α is the section at α of the projection F of E onto $\mathbb{R}^p \times \mathbb{R}^k$.

Proof. This follows from the description of these sets. Indeed, $\limsup E_t$ is equal to

$$\{x \mid \forall\varepsilon > 0,\ \forall\delta > 0,\ \exists t\colon 0 < \|t - t_0\| < \delta,\ \exists x' \in E_t\colon \|x' - x\| < \varepsilon\},$$

while $\liminf E_t$ coincides with the set

$$\{x \mid \forall\varepsilon > 0,\ \exists\delta > 0,\ \forall t \in F\colon 0 < \|t - t_0\| < \delta,\ \exists x' \in E_t\colon \|x' - x\| < \varepsilon\}.$$

Thus both are described by first order formulae which accounts for their definability — cf. [4].[9] □

Remark 2.6. It is important to note that the last statement does not hold unless the projection $F = \pi_k(E)$ is definable (meaning the family $\{E_t\}_{t\in F}$ is definably parametrised), i.e. it is no longer true for sequences $\{E_\nu\}_{\nu=1}^{+\infty}$ of definable subsets of $\mathbb{R}^n$. Just think of the Weierstrass Approximation Theorem.

This theorem was first obtained for Hausdorff limits of semialgebraic families by L. Bröcker in [3] (the approach is somewhat different).

Let us note again some general facts:

Proposition 2.7. *Let E be a closed subset of $\mathbb{R}^k_t \times \mathbb{R}^m_x$. Then*:

(1) $\{0\} \times \limsup E_t = \overline{E \setminus (\{0\} \times E_0)} \cap \{t = 0\} \subset \{0\} \times E_0$.
(2) $\liminf E_t \subset E_0$.

Proof. The condition $x \in \limsup E_t$ is equivalent to the following: for each neighbourhood V_0 of $0 \in \mathbb{R}^k$ and each neighbourhood W_x of $x \in \mathbb{R}^m$, the set $V_0 \times W_x \cap (E \setminus (\{0\} \times E_0))$ is nonempty. This amounts to saying that $(0, x) \in \overline{E \setminus (\{0\} \times E_0)}$ and therefore we obtain the equality sought after.

As E is closed, obviously $\overline{E \setminus (\{0\} \times E_0)} \subset E$ and so we have also the inclusion wanted.

The second part follows from the fact that if $(0, x_0) \notin E$, then there is a neighbourhood $V \times U$ of this point disjoint with E. Thus for any $t \in V$, $E_t \cap U = \varnothing$. □

Remark 2.8. It follows from (1) that $\limsup E_t$ coincides with what L. Bröcker denotes $\lim_{F\to 0} E_F$ in the semialgebraic case (see [3], §2.1).

Corollary 2.9. *Let E be a closed subset of $\mathbb{R}^k \times \mathbb{R}^m$. Then*:

(1) $E_0 = \limsup E_t$ *iff* $\overline{E \setminus (\{0\} \times E_0)} = E$.
(2) $E = \lim E_t$ *iff* $E_0 \subset \liminf E_t$.

We go back now to the definable case. Recall that in this case by the main result of [5], we have the following characterisation: $\limsup E_t = E_0$ if and only if $\forall x \in E_0$ there is $\dim_x E_0 < \dim_{(0,x)} E$, where the dimension of a definable set $G \subset \mathbb{R}^n$ at a point $x \in \overline{G}$ is by definition

$$\dim_x G = \min\{\dim(G \cap U) \mid U \text{ a definable neighbourhood of } x\}$$

with $\dim(G \cap U)$ understood as the maximal dimension of a definable cell contained in $G \cap U$ (see [4]).

Moreover, recall that not all subanalytic subsets of $\mathbb{R}^m$ are definable in some o-minimal structure. Therefore, in some of our theorems we need the notion of relative compactness in one direction,[10] introduced by Łojasiewicz.

Lemma 2.10. *Let $E \subset \mathbb{R}^k_t \times \mathbb{R}^n_x$ be a closed and definable (or x-relatively compact subanalytic) set. Then the function*

$$\delta : E \ni (t, x) \mapsto \dim_x E_t \in \mathbb{Z}_+$$

is definable.

Proof. In the subanalytic case we proceed as in the definable one below, after having replaced the set E by $E \cap ([-m, m]^k \times \mathbb{R}^n)$, $m \in \mathbb{N}$. Once we have the subanalycity of δ restricted to $[-m, m]^k$ with arbitrary m, we conclude that it is subanalytic. Therefore, we assume hereafter that E is definable.

[9] E.g. condition $\|x' - x\| < \varepsilon$ is equivalent to the polynomial one $\sum_1^n (x'_j - x_j)^2 < \varepsilon^2$, while condition $\exists x' \in E_t$ is to be understood as $\exists x' \in \mathbb{R}^n\colon (x', t) \in E$.

[10] A set $E \subset \mathbb{R}^k_t \times \mathbb{R}^n_x$ is x-relatively compact if for any relatively compact set $K \Subset \mathbb{R}^k$ and the projection $\pi_k(t, x) = t$, the set $\pi_k^{-1}(K) \cap E$ is relatively compact. In other words, the restriction of π_k to $\overline{E}$ is proper.

The function δ having discrete values and being bounded (by $\dim E$), we need only to show the definability of its fibres. This follows from definable cell decomposition.[11] Indeed, take a definable cell decomposition $\mathcal{C}$ of $\mathbb{R}^k \times \mathbb{R}^n$ compatible with E.[12] Let $\pi(t,x) = t$. We recall that (cf. [4], Chapter 3):

(1) $\mathcal{C}' := \{\pi(C) \mid C \in \mathcal{C}\}$ forms a cell decomposition of $\mathbb{R}^k$ compatible with $F := \pi(E)$.
(2) For each $t \in \mathbb{R}^k$, the sections C_t of cells $C \in \mathcal{C}$ projecting onto the unique cell $C' \subset \mathbb{R}^k$ containing t, form a cell decomposition of $\mathbb{R}^n$ compatible with E_t.

Fix $t \in F$ and consider those cells $C_1, \dots, C_r \in \mathcal{C}$ for which $C_j \subset E$ and $\pi(C_j) = C'$ where $C' \in \mathcal{C}'$ is the unique cell containing t. Then $E_t = \bigcup (C_j)_t$ and we have $\dim_x E_t = \max \dim_x (C_j)_t$, where we put $\dim_x (C_j)_t := -1$, if x does not belong to the closure of $(C_j)_t$. However, a cell has constant dimension and so

$$\dim_x E_t = \max\bigl\{\dim (C_j)_t \bigm| j\colon x \in \overline{(C_j)_t}\bigr\}.$$

But then $\dim (C_j)_t = \dim C_j - \dim C'$. Therefore,

$$\dim_x E_t = \max\bigl\{\dim C_j \bigm| j\colon x \in \overline{(C_j)_t}\bigr\} - \dim C'. \tag{\#}$$

This implies that the set $\delta^{-1}(k) \cap (C' \times \mathbb{R}^n)$ coincides with

$$\bigcup \bigl\{\overline{C_j} \cap \bigl(C' \times \mathbb{R}^n\bigr) \bigm| j\colon \dim C_j = \dim C' + k\bigr\}$$

which is clearly a definable set. The lemma follows. $\square$

Theorem 2.11. *In the setting of the preceding lemma, $F \subset \mathbb{R}^k$ being the projection of E, both sets*

$$Ls(E) := \Bigl\{t_0 \in F\colon E_{t_0} = \limsup_{t \to t_0} E_t\Bigr\}, \qquad Li(E) := \Bigl\{t_0 \in F\colon E_{t_0} = \liminf_{t \to t_0} E_t\Bigr\}$$

are definable too, and so is the set $L(E) := \{t_0 \in F\colon E_{t_0} = \lim_{t \to t_0} E_t\}$ as their intersection.

Moreover, $L(E) = Li(E)$ and if only $\dim F > 0$, then one has also $\dim F \setminus L(E) < \dim F$ and of course $\dim F \setminus Ls(E) < \dim F$.

Proof. By the main result of [5], the set $Ls(E)$ can be described as

$$\{t \in F \mid \forall x \in E_t,\ \dim_x E_t < \dim_{(t,x)} E\}.$$

This is clearly the complement of the projection onto $\mathbb{R}^k$ of the set

$$\bigl\{(t,x) \in E \bigm| \dim_x E_t \geqslant \dim_{(t,x)} E\bigr\}.$$

The latter is definable due to the definability of the functions

$$(t,x) \mapsto \dim_x E_t \quad \text{and} \quad (t,x) \mapsto \dim_{(t,x)} E,$$

which for the first function follows from the previous lemma and is a classical fact for the second one.

Fix a cell decomposition of E. If $d := \dim F > 0$, then E contains a cell defined over a cell $C' \subset F$ of dimension d. From the formula (#) in the proof of Lemma 2.10 we know that for any $t \in C'$, $x \in E_t$, the dimension $\dim_x E_t = \max\{\dim C - d \mid C \subset E\colon \pi(C) = C',\ x \in \overline{C_t}\}$. On the other hand, $\dim_{(t,x)} E = \max\{\dim C \mid C \subset E\colon (t,x) \in \overline{C}\}$. This shows that $Ls(E) \neq \varnothing$. Besides, since this works for any cell $C' \subset F$ of dimension d, one obtains $\dim F \setminus Ls(E) < \dim F$.

The set $Li(E)$ is described by the inclusion $E_{t_0} \subset \liminf_{t \to t_0} E_t$ which by Corollary 2.9 means that it coincides with the set $L(E)$. As to its definability, it follows from the description using first order formulae

$$Li(E) = \bigl\{t_0 \in F \bigm| \forall x \in E_{t_0},\ \forall \varepsilon > 0,\ \exists \delta > 0\colon \forall t \in F\colon 0 < \|t - t_0\| < \delta,\ \exists x' \in E_t\colon \bigl\|x - x'\bigr\| < \varepsilon\bigr\}.$$

In order to show that $\dim F \setminus L(E) < \dim F$ one fixes as earlier a cell $C' \subset F$ of dimension $d = \dim F > 0$ and takes all the cells $C_j \subset E$ projecting onto C'. It is easy to see that the sections of a cell are continuous. Since $E_t = \bigcup (C_j)_t$ for $t \in C'$ and the Kuratowski convergence is continuous with respect to finite unions,[13] we get continuity of the sections of E along C'. As we have chosen a cell C' of maximal dimension, it follows that for any $t \in C'$ there exists a $\delta > 0$ such that the ball $\mathbb{B}(t,\delta)$ does not meet any other cell in F.[14] This implies that $C' \subset L(E)$. $\square$

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[11] See Section 4 for the notion of a definable cell and [4] for cell decomposition — a special type of stratification.

[12] Meaning that E is the disjoint union of some cells from $\mathcal{C}$.

[13] For a list of basic properties of the Kuratowski convergence see e.g. [7].

[14] Otherwise, in view of the finiteness of the cell decomposition of F, one would find a point $t_0 \in C'$ and a cell $C'' \subset F \setminus C'$ containing a sequence of points $C'' \ni t_\nu \to t_0$. But that would mean that $\overline{C''} \cap C' \neq \varnothing$, which implies $C' \subset \overline{C''} \setminus C''$ — the cell decomposition being a stratification — and so $d = \dim C' < \dim C'' \leqslant d$ which is impossible.

Remark 2.12. Observe that if F consists of isolated points, then according to our definitions of the upper and lower limits, there is $Ls(E) = Li(E) = \varnothing$.

Note that this theorem may be treated as an answer to L. Bröcker's Remark 3.5 in [3]. Compare also [10] and [1], Theorem 1.4.13.

From now on, we consider the following situation. Let $E \subset \mathbb{R}^k_t \times \mathbb{R}^m_x$ be a closed definable (respectively subanalytic and bounded) set, $0 \in F := \pi(E)$, where $\pi(t,x) = t$ is the natural projection. To shorten the notations we shall denote by $cc(E_t)$ the set of connected components of the section E_t. Any $S \in cc(E_t)$ is definable (resp. subanalytic), open and closed in E_t, and there is a common bound $\# cc(E_t) \leqslant N$ for all $t \in F$.[15]

We will assume hereafter that $E_0 = \lim E_t$ (this situation can be obtained in a more general definable situation using the main result of [10], see also Theorem 2.11). To understand better the difference between the Kuratowski limit and upper limit consider

Example 2.13. Take the closure E of the graph of $x = \operatorname{Arg} t$ (argument of t) for $t \in \mathbb{C} \cong \mathbb{R}^2$ with $\operatorname{Arg} t \in [0, \pi/2]$. In this case we have $E_0 = \limsup E_t$, while $\liminf E_t = \varnothing$. Observe also that each section E_t is connected, but E_0 has dimension 1 while the other sections are zero-dimensional.

An example of a set E satisfying the assumption $E_0 = \lim E_t$ is provided e.g. by any finite union of graphs of continuous functions $f_j : F_j \to \mathbb{R}^k_t$ which are open onto their images, where $F_j \subset \mathbb{R}^n_x$ are closed sets. Indeed, continuity together with the openness on $f_j(F_j)$ implies that $f_j^{-1}(t)$ converges to $f_j^{-1}(0)$ in the sense of Kuratowski, while each E_t is the union of such level sets.[16]

The question we are first dealing with is: under what kind of assumptions on E, k or m the following two conditions hold true:

(1) $\# cc(E_0) \leqslant \# cc(E_t)$ for all $t \in F$ belonging to some neighbourhood of zero.
(2) For any $S \in cc(E_0)$ there is a neighbourhood of zero in F such that for any t from this neighbourhood one can find a collection $\{S^t_1, \ldots, S^t_{r_t}\} \subset cc(E_t)$ for which $S = \lim \bigcup_j S^t_j$.

Note that in the previous example (the argument's graph) only (1) holds.

Example 2.14. In the example from the previous section we have seen that property (1) need not be true for an arbitrary compact set. One can also have $\# cc(E_0) = 1$ while $\# cc(E_t) = \infty$ for all $t \neq 0$ — it suffices to take

$$E := \bigcup_n \bigl\{(t,x) \in \mathbb{R}^2 \bigm| 1/(n+1) + t^n \leqslant x \leqslant 1/n + t^n,\ t \in [0,1]\bigr\}.$$

Another example for which $\# cc(E_t) < \# cc(E_0) < +\infty$ whenever $t \in F \setminus \{0\}$ is given by the set

$$E := \bigl[\{0\} \times \bigl(\{0\} \cup \{1\}\bigr)\bigr] \cup \left(\bigcup_{\nu=1}^{+\infty} \left\{\frac{1}{2\nu}\right\} \times \{0\}\right) \cup \left(\bigcup_{\nu=1}^{+\infty} \left\{\frac{1}{2\nu - 1}\right\} \times \{1\}\right).$$

Note also that compactness is necessary even in the semialgebraic case: let E be the closure of

$$\bigl\{(t,x,y) \in (0,1] \times \mathbb{R} \times \mathbb{R} \bigm| ty = x^2 - 1\bigr\}.$$

Here $\# cc(E_0) = 2$ while $\# cc(E_t) = 1$ for $t \neq 0$ and moreover $E_0 = \lim E_t$.

Finally, a most simple example shows why we should work with the limit instead of the upper limit: for the set

$$E := \bigl([-1,0] \times \{-1,1\}\bigr) \cup \bigl([0,1] \times \{1\}\bigr) \subset \mathbb{R}_t \times \mathbb{R}_x$$

condition (1) is not satisfied and yet $E_0 = \limsup E_t$.

Example 2.15. To obtain (2) one has to consider the union of some connected components rather than single components even in the semialgebraic case. Indeed, consider

$$E := \bigl\{(t,x) \in [0,1] \times \mathbb{R} \bigm| t \leqslant x \leqslant 1 + t \text{ or } -t-1 \leqslant x \leqslant -t\bigr\}.$$

[15] It is nowadays a classical result — due to Gabrielov in the subanalytic setting; a different proof given by Łojasiewicz's group is to be found in [6]; for the definable version see [4].

[16] The operation of taking finite unions is continuous with respect to the Kuratowski convergence.

Nonetheless, for the upper limit even in the semi-analytic compact case there may be a problem with (2), if the space of parameters is too ‘big’:

Example 2.16. Let

$$E := \left(\left\{(t_1, t_2) \in [-1, 1]^2 \;\middle|\; t_1 = t_2\right\} \times [0, 1]\right) \cup \left(\left\{(t_1, t_2) \in [-1, 1]^2 \;\middle|\; t_1 = -t_2\right\} \times [-1, 0]\right) \subset \mathbb{R}^2_t \times \mathbb{R}_x.$$

Note that same set seen in $\mathbb{R}^3$ as in $\mathbb{R}_t \times \mathbb{R}^2_x$ fulfils (2). Even if we assume that $0 \in \operatorname{int} F$, (2) may fail to hold:

$$E := \left([-1, 1]^2 \times \{0\}\right) \cup \left(\{0\} \times [0, 1]^2\right) \cup \left(\{0\} \times [-1, 0]^2\right) \subset \mathbb{R}^2_t \times \mathbb{R}_x;$$

another type of example in $\mathbb{R}^2_t \times \mathbb{R}_x$ is

$$E := \left([-1, 1]^2 \times \{0\}\right) \cup \left([0, 1] \times [-1, 1] \times [0, 1]\right) \cup \left([-1, 0] \times [-1, 1] \times [-1, 0]\right).$$

The main problem which appears in the general case is the following: any $x_0 \in E_0$ is the limit of a subnet $x_{t_\alpha} \in E_{t_\alpha}$ and the points from any neighbourhood of x_0 must be approximable by points from the connected components of the points x_{t_α} — they cannot simply spring out from nowhere. However, their respective neighbourhoods of convergence may be decreasing with no control over them. This kind of phenomenon is impossible in the definable setting.

3. On the connected components of the sections

We begin with the following theorem. Throughout this section we put $F := \pi(E)$, where $\pi(t, x) = \pi_k(t, x) = t$ and E is as follows.

Theorem 3.1. *Let $E \subset \mathbb{R}^k_t \times \mathbb{R}^n_x$ be a definable (respectively x-relatively compact subanalytic) set. Then the function*

$$\nu : \mathbb{R}^k \ni t \mapsto \#cc(E_t) \in \mathbb{Z}_+$$

is definable (respectively subanalytic).

In order to prove this theorem we shall need the following notion:

Definition 3.2. Let $C, D \subset \mathbb{R}^m$ be two definable cells.[17] We say that C is *adjacent to* D, if $C \cap \overline{D} \neq 0$. We write then $C \prec D$.

Remark 3.3. The relation is not symmetric (think of $C = \{0\}$ and $D = (0, 1)$[18]), but it has the property that $C \prec D$ implies that $C \cup D$ is (pathwise) connected.[19]

Lemma 3.4. *Let $C, D \subset \mathbb{R}^k_t \times \mathbb{R}^n_x$ be two (disjoint) definable cells with $\pi_k(C) = \pi_k(D) =: G$. Then the set*

$$G_0 := \{t \in G \mid C_t \prec D_t\}$$

is definable.

Proof. It follows directly from a description of G_0 by a first order formula:

$$G_0 = \left\{t \in G \;\middle|\; \exists x \in \mathbb{R}^n\colon\ (t, x) \in C \text{ and } \operatorname{dist}(x, D_t) = 0\right\}$$

and the fact that the distance function $(t, x) \mapsto \operatorname{dist}(x, D_t)$ is definable (which is a classical result for the Euclidean distance following from the description of its graph by a first order formula). □

Proof of Theorem 3.1. Just as in the proof of Lemma 2.10 we may assume hereafter that E is definable.

As already noted, there exists an $N \in \mathbb{N}$ such that $\nu(t) \leqslant N$ for all $t \in \mathbb{R}^k$. The function ν having discrete values, its definability (i.e. the definability of its graph) is equivalent to the definability of its fibres $\nu^{-1}(m)$, $m \in \{1, \dots, N\}$. In other words, we want to find a cell decomposition of $\mathbb{R}^k$ (actually, any finite decomposition into disjoint definable cells will suffice) such that ν is constant on each cell.

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[17] See Section 4.

[18] In general, if the cells are disjoint, they cannot be both adjacent one to another at the same time; more precisely, $C \cap D = \varnothing$ and $C \prec D$ implies $C \subset \overline{D} \setminus D$.

[19] It follows from the Curve Selecting Lemma and the fact that any cell is (definably) pathwise connected.

Take a definable cell decomposition $\mathcal{C}$ of $\mathbb{R}^k \times \mathbb{R}^n$ compatible with E. We keep the notations introduced in the proof of Lemma 2.10: $\mathcal{C}' := \{\pi(C) \mid C \in \mathcal{C}\}$.

Fix a cell $D \in \mathcal{C}'$ and put

$$\mathcal{C}_D := \{C \in \mathcal{C} \mid \pi(C) = D \text{ and } C \subset E\}.$$

According to Lemma 3.4 each pair $(C_1, C_2) \in \mathcal{C}_D^2$ yields a definable set (maybe empty)

$$D(C_1, C_2) := \{t \in D \mid (C_1)_t \prec (C_2)_t\}.$$

Therefore, we may decompose $D = \bigcup D_i$ into a finite number of cells compatible with all these sets $D(C_1, C_2)$. For each $C \in \mathcal{C}_D$ we obtain a decomposition into pairwise disjoint cells $C = \bigcup [C \cap (D_i \times \mathbb{R}^n)]$.

Fix D_i and put

$$\mathcal{C}_{D_i} := \{C \cap (D_i \times \mathbb{R}^n) \mid C \in \mathcal{C}_D\}.$$

Then, for any pair of cells $(C_1, C_2) \in \mathcal{C}_{D_i}^2$ we have either

$$\forall t \in D_i, \quad (C_1)_t \cap \overline{(C_2)_t} = \varnothing, \tag{1}$$

or

$$\forall t \in D_i, \quad (C_1)_t \cap \overline{(C_2)_t} \neq \varnothing, \tag{2}$$

the latter being equivalent to C_1 meeting the closure of C_2 in $D_i \times \mathbb{R}^n$. In particular, if the condition (2) occurs, then $C_1 \cup C_2$ has connected sections over D_i.

Finally, we consider the equivalence relation defined on $\mathcal{C}_{D_i}$ by

$$C_1 \sim C_2 \iff \exists K_1, \ldots, K_r \in \mathcal{C}_{D_i}\colon\ C_1 * K_1 * \cdots * K_r * C_2,$$

where each $*$ stands either for $\prec$, or for $\succ$.[20] Given an equivalence class $[C]$, we put $\hat{C} := \bigcup\{K \mid K \in [C]\}$. In this way we get a finite number of definable sets $\hat{C}_1, \ldots, \hat{C}_r$ which are pairwise separated in $D_i \times \mathbb{R}^n$,[21] have connected sections and $\bigcup \hat{C}_j = E \cap (D_i \times \mathbb{R}^n)$. Since these properties hold also section-wise over D_i, it follows easily that for each $t \in D_i$, the r sets $(\hat{C}_j)_t$ form the family $cc(E_t)$ and the theorem is proved. □

Remark 3.5. It follows from the proof that the equivalence relation defined on E by

$$(t, x)\mathcal{R}(t', x') \iff t = t' \text{ and } x, x' \text{ belong to the same } S \in cc(E_t)$$

is definable (i.e. has a definable graph). Observe that $[(t, x)]$ is exactly $\{t\}$ times the connected component of E_t containing x.

First we will consider a one-dimensional parameter (i.e. $k = 1$), as this situation is rather special. Besides, it seems interesting to consider it on its own in the definable setting (though Theorem 3.7 hereafter follows from Theorem 3.9).

Proposition 3.6. *If $E \subset \mathbb{R} \times \mathbb{R}^n$ is definable/subanalytic and bounded, then there exists $\varepsilon > 0$ such that ν is constant on the intervals $(0, \varepsilon)$ and $(-\varepsilon, 0)$.*

Proof. Take a stratification of $\mathbb{R}$ compatible with the sets $\{\nu = k\}$, $k = 0, 1, \ldots, N$, where N is the bound for ν. The strata are either intervals, or points. The point 0 is in the stratum Γ_0 which is either 0, or an open interval containing zero. In the first case the two adjacent strata are of the form $(0, \delta)$ and $(-\gamma, 0)$ with $\delta, \gamma > 0$, and ν is constant on either side of zero. In the second case ν is constant in a neighbourhood of zero. □

Theorem 3.7. *If $E \subset \mathbb{R} \times \mathbb{R}^n$ is a compact definable/subanalytic set and $E_0 = \lim E_t$ (either when $t \to 0$ or just $t \to 0^+$), then $\# cc(E_t) \geqslant \# cc(E_0)$ for all t small enough. Moreover, each $S \in cc(E_0)$ is the limit of unions of certain connected components of the sections E_t: there is a definable set $E^S \subset E$ such that for all $t \in F$ in a neighbourhood of zero, $\varnothing \neq cc(E_t^S) \subset cc(E_t)$ and $E_t^S \to S$.*

Proof. The first part, i.e. $\nu(t) \geqslant \nu(0)$, is obvious, since ν happens to be constant on either side of zero (cf. the previous Proposition) and $E_0 = \lim E_t$.

[20] For instance, $\{0\} \prec (0, 1) \succ \{1\}$.

[21] A and B are separated in a topological space X if $\overline{A} \cap B = \varnothing$ and $\overline{B} \cap A = \varnothing$.

Let $\pi : \mathbb{R}_t \times \mathbb{R}^n_x \to \mathbb{R}_t$ be the natural projection. Fix a connected component $S \subset \{0\} \times E_0$ and any point $z_0 \in S$.[22] By the Curve Selecting Lemma there is a definable (respectively subanalytic) continuous function $\gamma : [0,1] \to \mathbb{R} \times \mathbb{R}^n$ such that $\gamma((0,1]) \subset E \setminus (\{0\} \times E_0)$ and $\gamma(0) = z_0$. Thus $\pi(\gamma(\tau)) \neq 0$ for $\tau \in (0,1]$ and $\pi(\gamma(0)) = 0$. Observe that if zero lies in the interior of $\pi(E)$, then we can choose 'the side' over which γ is defined, i.e. we can ask that $\pi(\gamma([0,1])) \subset \mathbb{R}_+$.[23] In particular we may assume that $\pi \circ \gamma$ is increasing and $\rho(\gamma([0,1])) \subset U$ where $\rho(t,x) = x$ and U is a neighbourhood of $\rho(S)$ separating it from the other connected components of E_0. We may also assume that $\pi(\gamma([0,1])) = [0,\varepsilon]$. Taking a reparametrisation, we may assume that γ is defined of $[0,\varepsilon]$ and $(\pi \circ \gamma)(t) = t$.

For each $t \in (0,\varepsilon]$ let D_t denote the connected component of the set $\{t\} \times E_t$ containing $\gamma(t)$ and put

$$H := \bigcup_{t \in (0,\varepsilon]} D_t \subset \mathbb{R} \times \mathbb{R}^n.$$

Note that $z_0 \in \liminf H_t$.

We now prove that $\overline{H} \cap \{t = 0\} \subset S$.[24] Suppose that $z \in \overline{H} \cap \{t = 0\} \setminus S$. Clearly there must exist another connected component $S' \subset \{0\} \times E_0$ for which $z \in S'$. Take a sequence $\{z^\kappa\} \subset H$ such that $z^\kappa \to z$ and fix a neighbourhood $U' \supset \rho(S')$ disjoint with U and the other components of E_0. Without loss of generality we may assume that $\pi(z^\kappa) \in (0,\varepsilon]$ and $\rho(z^\kappa) \in U'$ for all indices κ.

For each κ, let $L^\kappa := H \cap \{t = \pi(z^\kappa)\}$. Observe that the sets $(\mathbb{R} \times U) \cap L^\kappa$ and $(\mathbb{R} \times (\mathbb{R}^n \setminus \overline{U})) \cap L^\kappa$ are disjoint, nonempty (since for some $t \in (0,\varepsilon]$, $\gamma(t)$ belongs to one of these sets, while z^κ to the other) and open in L^κ. By the connectedness of L^κ we have $(\mathbb{R} \times (\overline{U} \setminus U)) \cap L^\kappa \neq \varnothing$ and so this set contains a point w^κ. Extracting, if necessary, a subsequence, we may assume that $w^\kappa \to w$, where w is a point from $\{0\} \times \mathbb{R}^n$. Obviously $w \in \{0\} \times (\overline{U} \setminus U)$, but that means w cannot belong to $\{0\} \times E_0$ which is a contradiction. We have thus proved that $\limsup H_t \subset S_0$.

Now, remark that if S' is, as above, another connected component of $E \cap \{t = 0\}$ and we construct similarly a set H', then for some $0 < \varepsilon' < \varepsilon$ the sets $H \cap \{t = s\}$ and $H' \cap \{t = s\}$ are disjoint when $s \in (0,\varepsilon']$. Indeed, if it were not the case, we would find a sequence $\{a^\kappa\} \subset H \cap H'$ such that $\pi(a^\kappa) \to 0$ and — extracting, if necessary, a subsequence — we would obtain $a^\kappa \to a \in \{0\} \times \mathbb{R}^n$. But then $a \in S \cap S'$ which is a contradiction.

It remains to prove that $\liminf H_t \supset S$ and that is where we must take into consideration unions.[25] Actually, it suffices to repeat the construction of $H = H(z_0)$ for each point of $z_0 \in S$. On the other hand, this construction done for the set $E^- := E \cap [(-\infty, 0] \times \mathbb{R}^n]$ produces a similar set $H'(z_0)$ projecting onto $\pi(E) \cap (-\infty, 0]$ (if the latter reduces to $\{0\}$, then we put $H'(z) := \varnothing$).

Let E^S be the closure of $\bigcup_{z \in S}[H(z) \cup H'(z)]$. Then by the preceding arguments we have $S_0 \subset \liminf E^S_t$ and $\limsup E^S_t \subset S_0$, which means that $S_0 = \lim E^S_t$. Besides, it is clear from the construction that for any $t \in F$ in a neighbourhood of zero, the E^S_t are unions of certain connected components of the E_t. Moreover, the set E^S is definable (respectively subanalytic). □

Remark that in this theorem we cannot replace the limit by the upper limit as was already shown in Example 2.14 (last example: for $t > 0$, the sections E_t have less connected components than E_0 and one of the components of E_0 is not 'attainable' from this side of the real axis).

Now we turn to proving an analogous result for multi-dimensional parameters. One cannot hope for a stabilisation of $\nu(t)$ near zero unless the parameters are one-dimensional. Indeed, consider the set $E = \{(t_1, t_2, x) \in \mathbb{R}^2 \times \mathbb{R} \mid t_2^2 = x^2\}$.

The result presented below does not need assuming any sort of definability. However, the finitude assumption on $\# cc(E_0)$ is necessary — cf. the following example:

Example 3.8. In our problem definability seems at first unavoidable with regard to the set $E \subset \mathbb{R}^2$ constructed as follows. Take the segment $[0,1] \times \{1\}$ and join its middle with the point $\{(0, 1/2)\}$ by another segment. Then join its middle with $\{(0, 1/2^2)\}$ and so on. The closure gives the set E. The section E_0 has infinitely many connected components, while each section E_t for $t \in (0,1]$ consists of only finitely many points (whose number is growing when we approach $t = 0$).

The definability of E in the general problem can be replaced by the assumption that E_0 has only finitely many connected components. Observe that this assumption is of course satisfied in the definable setting (or in the subanalytic compact setting).

Theorem 3.9. *Assume that $E \subset \mathbb{R}^k \times \mathbb{R}^n$ is compact with $E_0 = \lim E_t$ and $\# cc(E_0) < +\infty$. Then:*

(1) *$\# cc(E_t) \geqslant \# cc(E_0)$ for $t \in F$ in a neighbourhood of zero.*

[22] Note that S_0 is the corresponding connected component of E_0.

[23] To obtain this it suffices to replace at the beginning E with $E^+ := E \cap ([0, +\infty) \times \mathbb{R}^n)$.

[24] For a fixed $t_0 \in \mathbb{R}$, we write $\{t = t_0\} := \{t_0\} \times \mathbb{R}^n$.

[25] Think of the first example from 2.15.

(2) *For each $S \in cc(E_0)$ there exists a compact set $E^S \subset E$ such that $\varnothing \neq cc(E_t^S) \subset cc(E_t)$ for all $t \in F$ in a neighbourhood of zero and $S = \lim E_t^S$. Moreover, if E is definable/subanalytic, then E^S is definable/subanalytic too with $\dim E^S > \dim S$.*

Proof. Write $cc(E_0) = \{S_1, \dots, S_r\}$, where $r = \# cc(E_0)$. Let us take an $\varepsilon > 0$ such that the ε-neighbourhoods $U_j^\varepsilon \supset S_j$ are pairwise disjoint ($U_j^\varepsilon := \bigcup_{a \in S_j} \mathbb{B}(a, \varepsilon)$). Let K_j be the closure of $U_j^\varepsilon \setminus U_j^{\varepsilon/2}$. These are pairwise disjoint, compact sets.

Since $K_j \cap E_0 = \varnothing$ and $E_0 = \limsup E_t$, for each $j = 1, \dots, r$, we can find a neighbourhood $\Omega_j \supset K_j$ and a neighbourhood V_j of $t = 0$ such that $E_t \cap \Omega_j = \varnothing$ for $t \in V_j \cap F$. Put $\Omega := \bigcup_1^r \Omega_j$ and $V := \bigcap_1^r V_j$. We have now $E \cap (V \times \Omega) = \varnothing$.

Fix $t \in V \cap F \setminus \{0\}$ and $S \in cc(E_t)$. Since by construction $S \cap \Omega = \varnothing$, then either there exists a j for which $S \subset U_j^\varepsilon$, or $S \subset \mathbb{R}^n \setminus U$, where $U := \bigcup_1^r U_j^\varepsilon$ (due to the connectedness of S). However, as $E_0 = \liminf E_t$, for all $t \in F$ in a neighbourhood of zero (which we may assume to be V), $E_t \cap U \neq \varnothing$. We have thus $S \cap \Omega = \varnothing$ and $S \cap U \neq \varnothing$, and so there must be $S \subset U_j^\varepsilon$ for some j.

Once we have established this fact, we observe that it implies (1). We turn now to proving (2). Keeping the notations from the first part of the proof we fix S_j and put $E^j := E \cap (V \times U_j^\varepsilon)$. Since by construction $cc(E_t^j) \subset cc(E_t)$ we obtain that E^j is compact. Note that in the definable/subanalytic setting we may ask that V be definable/subanalytic (U_j^ε are definable/subanalytic automatically) and so E^j is definable/subanalytic too. It remains to prove that $S_j = \lim E_t^j$.

To do this observe that by the assumptions each point $x \in S_j$ is the limit of a net $x_t \in E_t$. But $x_t \in U_j^\varepsilon$ for t close to zero, and so there must be $x_t \in E_t^j$. On the other hand, if we take a convergent subnet $x_{t_s} \to x$ with $x_{t_s} \in E_{t_s}^j$, then $x \in E_0$, because $E_{t_s}^j \subset E_{t_s}$. By construction $x \in S_j$.

The assertion concerning the dimension in the definable/subanalytic setting follows from the main result from [5]. Indeed, by the preceding proof, $S_j = \lim E_t^j = \limsup E_t^j$ and so for any $x \in S_j$ we have by [5], Theorem (3.1), $\dim_x S_j < \dim_{(0,x)} E^j$. Take an $x \in S_j$ for which $\dim_x S_j = \dim S_j$ and observe that there is $\dim_{(0,x)} E^j \leqslant \dim E^j$. $\square$

Before proving the next result we shall need the following lemmata:

Lemma 3.10. *Let $E \subset \mathbb{R}_t^k \times \mathbb{R}_x^n$ definable (resp. subanalytic x-relatively compact). Then the function*

$$d : F \ni t \mapsto \dim E_t \in \mathbb{Z}_+$$

is definable/subanalytic.

Proof. It follows from [4], Theorem 3.18 in the definable setting.

That works also in the subanalytic case since E is bounded in x. In that case one can also argue in a different way: E is the union of subanalytic leaves Γ_j with constant rank of the projection $\pi|_{\Gamma_j}$. Then since $\dim(\Gamma_j)_t = \dim \Gamma_j - \operatorname{rk}\pi|_{\Gamma_j}$ (see e.g. [6], Proposition 4.2) and $\dim E_t = \max \dim(\Gamma_j)_t$, one obtains the subanalycity of the level sets $\{\dim E_t \geqslant r\}$ which implies d is subanalytic. $\square$

Remark 3.11. The assertion of the lemma is no longer true even in the semi-analytic case, if we drop the boundedness assumption. To see this consider

$$E = \big([0,1] \times \{0\}\big) \cup \left(\bigcup_{\nu=1}^{+\infty} \{1/\nu\} \times [2\nu, 2\nu + 1] \right).$$

Here E and F are semi-analytic, but $\{t \in F \mid \dim E_t = 1\} = \bigcup_\nu \{1/\nu\}$.

Lemma 3.12. *Let $E \subset \mathbb{R}_t^k \times \mathbb{R}_x^n$ definable (resp. subanalytic x-relatively compact) with $\dim E_t \leqslant r$ for all t. Then $\dim E \leqslant r + \dim F$ (recall that $F = \pi_k(E)$).*

Proof. It is a classical result due to the fact that one can decompose E into a finite number of leaves Γ_j with constant rank of the projection $\pi|_{\Gamma_j}$. In fact, in the general definable setting (which in this case encloses the subanalytic case too, $E \cap ([-m, m] \times \mathbb{R}^n)$ being bounded for any $m \in \mathbb{N}$), E is a finite sum of cells C and for them, $\dim C_t = \dim C - \dim \pi(C)$ (see [4], Proposition 3.11). Similarly, in the subanalytic case it is easy to check (see e.g. [6], Proposition 4.2) that for each j, $\dim(\Gamma_j)_t = \dim \Gamma_j - \operatorname{rk}\pi|_{\Gamma_j}$, whenever $t \in \pi(\Gamma_j)$. Now, since $\dim E_t = \max \dim(\Gamma_j)_t$, we obtain $\dim \Gamma_j \leqslant r + \max \operatorname{rk}\pi|_{\Gamma_i}$, for each j. The lemma follows (because $\dim F = \max \dim \pi(\Gamma_j) = \max \operatorname{rk}\pi|_{\Gamma_j}$). $\square$

Now, let us have a look at the behaviour of the dimension of converging sections (compare [3], Corollary 2.8):

Theorem 3.13. *Let $E \subset \mathbb{R}^k \times \mathbb{R}^n$ be definable/subanalytic and compact with $E_0 = \lim E_t$. Then $\dim E_t \geqslant \dim E_0$ for all $t \in F = \pi_k(E)$ sufficiently close to zero.*

Proof. The dimension of E_0 being the maximal dimension of the connected components of E_0, and in view of Theorem 3.9, we may assume without loss of generality that E_0 is connected. By the preceding lemma, the set $G := \{t \in F \mid \dim E_t < \dim E_0\}$ is definable/subanalytic. In order to prove the theorem it suffices to show that $0 \notin \overline{G}$. If, on the contrary, $0 \in \overline{G} \setminus G$, then by the Curve Selecting Lemma, there is a definable/semi-analytic curve $\gamma : [0,1] \to \mathbb{R}^k$ such that $\gamma((0,1]) \subset G$ and $\gamma(0) = 0$.

Let $E' := \bigcup\{\{t\} \times E_t \mid t \in \gamma([0,1])\} = E \cap [\gamma([0,1]) \times \mathbb{R}^n]$. It is clearly a compact definable/subanalytic set. Moreover, $E'_0 = \lim E'_t$ (due to Corollary 2.9 and the definition of the lower limit). Finally, put $E'' := E' \setminus (\{0\} \times E_0)$. By the definition of E', there is $\dim E''_t \leqslant \dim E_0 - 1$ for all $t \in \gamma((0,1]) = \pi(E'')$. But that implies (cf. Lemma 3.12)

$$\dim E'' \leqslant \dim E_0 - 1 + \dim \pi\left(E''\right) = \dim E_0 = \dim\bigl(\{0\} \times E_0\bigr) < \dim E'',$$

the latter inequality following from the fact that $\{0\} \times E_0 \subset \overline{E''} \setminus E''$ (due to the convergence). This contradiction finishes the proof. $\square$

An important observation is that in the theorem above the limit cannot be replaced by the upper limit (unless $k = n = 1$, obviously) nor is the converse of the theorem true, as is shown in the following examples:

Example 3.14. Let E be the closure of

$$\left\{\bigl((t_1, t_2), x\bigr) \in (0,1)^2 \times \mathbb{R} \;\middle|\; x = \frac{2t_1t_2}{t_1^2 + t_2^2}\right\}.$$

Then $E_0 = [0,1] = \limsup E_t$ while $\dim E_t = 0$ for $t \neq 0$.

If now $E = \{t^2 + x^2 = 1\} \cup \{(0,0)\}$, then $\dim E_0 = \dim E_t = 0$ for all $t \in [-1,1]$, but $E_0 \neq \limsup E_t$.

Note also that the definability of F is necessary — clearly, one has $\{0\} \times [0,1] = \lim E_\nu$ where $E_\nu = \{1/\nu\} \times \{q_1, \ldots, q_\nu\}$ with $[0,1] \cap \mathbb{Q} = \{q_1, q_2, \ldots\}$. Here all the sections are semialgebraic but their union is not.

4. Semialgebraic approximation and connected components

For the convenience of the reader we recall now the following notion: a set $C \subset \mathbb{R}^m$ is called a *definable/subanalytic cell* if:

(1) for $m = 1$, C is a point or an open, nonempty interval;
(2) for $m > 1$,
- either $C = f$ is the graph of a continuous, definable/subanalytic (in $\mathbb{R}^m$) function $f : C' \to \mathbb{R}$, where $C' \subset \mathbb{R}^{m-1}$ ($\mathbb{R}^{m-1}$ is the subspace of the first $m-1$ variables in $\mathbb{R}^m$) is a definable/subanalytic cell; such a cell we shall call *thin*;
- or $C = (f_1, f_2)$ is a definable/subanalytic *prism*, i.e. $(f_1, f_2) = \{(x,t) \in \mathbb{R}^{m-1} \times \mathbb{R} \mid x \in C',\ f_1(x) < t < f_2(x)\}$, where $C' \subset \mathbb{R}^{m-1}$ is a definable/subanalytic cell and both functions $f_j : C' \to \mathbb{R} \cup \{\pm\infty\}$ are continuous, definable/subanalytic and such that $f_1 < f_2$ on C' and each f_j either takes all values in $\mathbb{R}$, or is constant.

Note that if $C = f$ is a thin cell over a cell C', the function f need not have a continuation to the closure of C' unless $\dim C' = 1$ and f is bounded. To see this consider $f(x,y) = 2xy/(x^2 + y^2)$ over $C' = \{0 < y < 1 - x\}$.

We will give an elementary proof of a theorem on semialgebraic approximation of subanalytic sets in the spirit of [2]. Thanks to Theorem 3.9 any such an approximation is shown to be simultaneous, so to speak, in the sense that each connected component of the limit set is approximated by some connected components of the approximating semialgebraic sets.

Theorem 4.1. *Let $E \subset \mathbb{R}^m$ be a compact subanalytic set. Then there exists a sequence $\{A_\nu\}$ of semialgebraic sets such that*:

(1) $E = \lim A_\nu$;
(2) *for each $a \in E$ and any neighbourhood U of a one has for ν large enough,* $\dim U \cap E = \dim U \cap A_\nu$.

Moreover, for each such sequence $\{A_\nu\}$ one has the following: *for any $S \in cc(E)$ there is a sequence $\{S_\nu\}$ such that each S_ν is the union of some connected components of A_ν and* (1) *and* (2) *holds for S and the sequence $\{S_\nu\}$.*

Proof. Since E is compact, it admits a cell-decomposition into a finite number of bounded cells (see [4]) and so E is the union of their closures. Thus the problem reduces (by induction on m) to the problem of approximating a bounded subanalytic cell in $\mathbb{R}^2$ defined over an open interval (note that every cell in $\mathbb{R}$ is semialgebraic as well as every subanalytic cell in $\mathbb{R}^2$ defined over a point). This we obtain from the Stone–Weierstrass Approximation Theorem.

Let us be somewhat more precise, since there are several technical subtleties. We start induction on $m \geqslant 2$.[26]

(a) Let $m = 2$. Then the function $f : (a, b) \to \mathbb{R}$ defining a thin cell is subanalytic and bounded, whence it has a continuation $\tilde{f}$ to $[a, b]$, which is a compact interval. Therefore there exists a sequence of polynomials $\{P_\nu\}$ approximating f uniformly on $[a, b]$. This is equivalent to say that the graphs of $P_\nu|_{[a,b]}$ converge in the sense of Kuratowski to the graph of $\tilde{f}$ (over $[a, b]$) (cf. Proposition 1.1).

Now let us show how to approximate a subanalytic prism (f_1, f_2) over (a, b). One has to do with the fact that the polynomials P_ν and G_ν approximating on $[a, b]$ the functions $\tilde{f}_1$ and $\tilde{f}_2$ need not necessarily satisfy $P_\nu < G_\nu$ on $[a, b]$ (note that the function f_1, f_2 may have the same value at a or b). That is why we have first to separate f_1 and f_2 taking instead of them e.g. $f_{1,\nu} := \tilde{f}_1 - 1/\nu$ and $f_{2,\nu} := \tilde{f}_2 + 1/\nu$. For any $\nu \in \mathbb{N}$ let $p_\nu := P_\nu|_{[a,b]}$ and $g_\nu := G_\nu|_{[a,b]}$ be polynomials satisfying

$$\|p_\nu - f_{1,\nu}\|_{[a,b]} < 1/(2\nu) \quad \text{and} \quad \|g_\nu - f_{2,\nu}\|_{[a,b]} < 1/(2\nu),$$

where the norm is $\|h\|_K := \sup\{|h(x)|\colon x \in K\}$. We have $p_\nu \to \tilde{f}_1$ and $g_\nu \to \tilde{f}_2$ uniformly on $[a, b]$ and $p_\nu < g_\nu$ on this interval. It is clear that the compact semialgebraic prisms $\{(x, t)\colon x \in [a, b],\ p_\nu(x) \leqslant t \leqslant g_\nu(x)\}$ obtained in this way converge to the closure of (f_1, f_2) which is $\{(x, t)\colon x \in [a, b],\ \tilde{f}_1(x) \leqslant t \leqslant \tilde{f}_2(x)\}$.

Finally, since we approximate a given cell by a cell of the same type, the dimension is preserved (as it coincides with the maximal dimension of the cells of the decomposition).

(b) Fix $m > 2$ and assume that we already know how to approximate semialgebrically subanalytic cells in $\mathbb{R}^{m-1}$.

(i) First we deal with the case of a thin cell. Take $C = f$ to be the graph of a continuous, bounded subanalytic function $f : C' \to \mathbb{R}$, where $C' \subset \mathbb{R}^{m-1}$ is a bounded subanalytic cell. We may write $C' = \bigcup K_\nu$ where K_ν are subanalytic compact sets with $K_\nu \subset K_{\nu+1}$.[27] We may ask that $\dim K_\nu = \dim C'$ for all ν large enough. For each of these sets we apply the induction hypothesis to obtain a sequence $\{C'_\mu(K_\nu)\}_\mu$ of semialgebraic compact sets approximating K_ν and preserving the dimension. Observe that $\overline{C'} = \lim K_\nu$.

On the other hand, for each K_ν we can find a polynomial P_ν such that $\|P_\nu - f\|_{K_\nu} < 1/\nu$. Moreover, by the uniform continuity of P_ν on the compact set $\overline{C'}^{1/\nu} = \bigcup_{x' \in \overline{C'}} \overline{\mathbb{B}}(x', 1/\nu)$ we find a $\delta_\nu \in (0, 1/\nu)$ such that for any $x' \in \overline{C'}$ and any two points $y', y'' \in \overline{\mathbb{B}}(x', \delta_\nu)$ there is $|P_\nu(y') - P_\nu(y'')| < 1/\nu$.

Now, for each ν, let $C'_\nu := C'_\mu(K_\nu)$ be such that $\operatorname{dist}_H(C'_\mu(K_\nu), K_\nu) < \delta_\nu$ and put $C_\nu := P_\nu|_{C'_\nu}$. It is easy to check that the semialgebraic sets C'_ν converge to $\overline{C'}$. It remains to prove that $\lim C_\nu = \overline{C} = \overline{f}$.

First, we will show that $\liminf C_\nu \supset \overline{f}$. To this aim fix $x = (x', t) \in \overline{f}$. There is a sequence $f \ni (x'_\nu, t_\nu) \to (x', t)$, where $t_\nu = f(x'_\nu)$, $x'_\nu \in C'_\nu$. After passing to a subsequence and renumbering if necessary, we may assume that $x'_\nu \in K_\nu$ for each ν. By the choice of C'_ν there is in particular $K_\nu \subset C'_\nu + \overline{\mathbb{B}}(\delta_\nu)$ and so we find points $y'_\nu \in C'_\nu$ for which $\|y'_\nu - x'_\nu\| \leqslant \delta_\nu$. Since $\delta_\nu \to 0$, there is $y'_\nu \to x'$. But as $x'_\nu \in K_\nu$, we have on the one hand $|P_\nu(x'_\nu) - f(x'_\nu)| < 1/\nu$, while on the other the inequality satisfied by y'_ν implies $|P_\nu(x'_\nu) - P_\nu(y'_\nu)| < 1/\nu$. Therefore, $P_\nu(x'_\nu) \to t$ and so $C_\nu \ni (x'_\nu, P_\nu(x'_\nu)) \to x$.

We turn to proving the inclusion $\limsup C_\nu \subset \overline{f}$. Take any convergent subsequence $C_{\nu_k} \ni x_{\nu_k} \to x$, i.e. $x_{\nu_k} = (x'_{\nu_k}, P_{\nu_k}(x'_{\nu_k}))$, $x'_{\nu_k} \in C'_{\nu_k}$ and $x = (x', t)$. By the assumptions there must be $x' \in \overline{C'}$. Moreover, since $\operatorname{dist}_H(C'_{\nu_k}, K_{\nu_k}) < \delta_{\nu_k}$, there is in particular $C'_{\nu_k} \subset K_{\nu_k} + \overline{\mathbb{B}}(\delta_{\nu_k})$ and so we find points $y'_{\nu_k} \in K_{\nu_k}$ satisfying $\|x'_{\nu_k} - y'_{\nu_k}\| \leqslant \delta_{\nu_k}$. The sequence $\{y'_{\nu_k}\}_k$ clearly converges to x'. The inequality satisfied by these points, together with the fact that $y'_{\nu_k} \in K_{\nu_k}$, implies that on the one hand $|P_{\nu_k}(x'_{\nu_k}) - P_{\nu_k}(y'_{\nu_k})| < 1/\nu_k$, while on the other $|P_{\nu_k}(y'_{\nu_k}) - f(y'_{\nu_k})| < 1/\nu_k$. Therefore, $P_{\nu_k}(y'_{\nu_k}) \to t$ and thus $f(y'_{\nu_k}) \to t$. This means that $x \in \overline{f}$.

(ii) For a prism C the reasoning is similar though somewhat more technical. Indeed, suppose that $C = (f, g)$ with $f, g : C' \to \mathbb{R}$ continuous, subanalytic bounded functions defined on a bounded cell $C' \subset \mathbb{R}^{m-1}$ and satisfying $f < g$. Once again we construct the compacts K_ν and choose polynomials P_ν and Q_ν sufficiently close on K_ν to the functions $f - 1/\nu$ and $g + 1/\nu$ respectively. 'Sufficiently' means in this case that we ask that $P_\nu < Q_\nu$ on K_ν (which can be obtained, due to the fact that $f < g$ on K_ν, by taking P_ν and Q_ν such that $\|P_\nu - (f - 1/\nu)\|_{K_\nu} < 1/(2\nu)$ and $\|Q_\nu - (g + 1/\nu)\|_{K_\nu} < 1/(2\nu)$ as in the case $m = 1$). Then we attach to each set K_ν the number $\delta_\nu = \min\{\delta_\nu^{P_\nu}, \delta_\nu^{Q_\nu}\}$, where $\delta_\nu^{P\nu}$ and $\delta_\nu^{Q\nu}$ are found in $(0, 1/\nu)$ by the uniform continuity of P_ν, Q_ν on $\overline{C'}^{1/\nu}$ for $\varepsilon = \varepsilon_\nu = 1/(4\nu)$. That is, e.g. for P_ν

$$\forall x' \in \overline{C'}^{1/\nu},\ \forall y', y'' \in \overline{\mathbb{B}}(x', \delta_\nu), \quad \left|P_\nu(y') - P_\nu(y'')\right| < 1/(4\nu).$$

Finally, we choose C'_ν as earlier. Then $P_\nu < Q_\nu$ also on C'_ν and so we obtain a sequence of semialgebraic sets

$$C_\nu = [P_\nu|_{C'_\nu}, Q_\nu|_{C'_\nu}] := \left\{(x', t) \,\middle|\, x' \in C'_\nu,\ P_\nu(x') \leqslant t \leqslant Q_\nu(x')\right\}$$

converging to $\overline{(f, g)}$.

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[26] If $m = 1$ there is nothing to do, but it seems interesting, however, to see how the whole thing works in the case $m = 2$.

[27] If C' is not compact, this may be achieved for instance by taking $K_\nu := \{x' \in C' \mid \operatorname{dist}(x', \delta(C')) \geqslant \frac{1}{\nu}\}$ where $\delta(C') := \overline{C'} \setminus C'$.

Indeed, fix $(x',t)\in\overline{(f,g)}$. There is a sequence $(f,g)\ni(x'_\nu,t_\nu)\to(x',t)$ and we may ask (passing to a subsequence and renumbering if necessary) that $x'_\nu\in K_\nu$. We proceed as follows. For $x'_1\in K_1$ and $\varepsilon_1:=\min\{t_1-f(x'_1),g(x'_1)-t_1\}/2>0$ we find an index μ_1 and points $y'_\mu\in C'_\mu$ for $\mu\geqslant\mu_1$ such that there is

- $\|P_\mu-f\|_{K_1}<\varepsilon_1/2$ and $\|Q_\mu-g\|_{K_1}<\varepsilon_1/2$, $\mu\geqslant\mu_1$,
- $\|y'_\mu-x_1\|\leqslant\delta_\mu$ (note that $K_1\subset K_\mu$) and so we have the inequalities $|P_\mu(y'_\mu)-P_\mu(x_1)|<\varepsilon_1/2$, $|Q_\mu(y'_\mu)-Q_\mu(x_1)|<\varepsilon_1/2$.

In particular we have $P_\mu(y'_\mu)<t_1<Q_\mu(y'_\mu)$ for $\mu\geqslant\mu_1$. We apply this reasoning to $x_2\in K_2$ and t_2 (with the due changes, as for ε_2, e.g.) obtaining and index μ_2 for which we ask in addition $\mu_2>\mu_1$. In this way we obtain a sequence $\mu_1<\mu_2<\cdots$ such that

$$\left(y'_\mu,t_j\right)\in C_\mu\quad\text{and}\quad\left\|y'_\mu-x'_j\right\|\leqslant\delta_\mu,\quad\text{for }\mu_j\leqslant\mu<\mu_{j+1}.$$

Now, we may construct the sequence of points $(x''_\mu,t'_\mu)\in C_\mu$ converging to (x',t): for μ up to μ_1 we take any points, then, for $\mu_1\leqslant\mu<\mu_2$ we put $(x''_\mu,t'_\mu):=(y'_\mu,t_1)$, finally, for $\mu_2\leqslant\mu<\mu_3$, $(x''_\mu,t'_\mu):=(y'_\mu,t_2)$ and so on. Clearly, $(x''_\mu,t'_\mu)\to(x',t)$. Thus $\overline{(f,g)}\subset\liminf C_\mu$.

Next, if $C_{\nu_k}\ni(x'_{\nu_k},t_{\nu_k})\to(x',t)$, then $x'\in\overline{C'}$. Moreover, there are points $y'_k\in K_{\nu_k}$ for which $\|x'_{\nu_k}-y'_k\|\leqslant\delta_{\nu_k}$ and so $y'_k\to x'$. We have $|P_{\nu_k}(x'_{\nu_k})-P_{\nu_k}(y'_k)|<1/(4\nu_k)$ and $|P_{\nu_k}(y'_k)-f(y'_{\nu_k})+1/\nu_k|<1/(2\nu_k)$; similar inequalities hold for Q_{ν_k} and $g+1/\nu_k$. We are looking for a sequence $(x''_k,t'_k)\in(f,g)$ converging to (x',t).

Either $f(y'_k)<t_{\nu_k}<g(y'_k)$, in which case we put $(x''_k,t'_k):=(y'_k,t_{\nu_k})\in(f,g)$, or $t_{\nu_k}\notin(f(y'_k),g(y'_k))$. If the latter occurs, then for $d_k:=\min\{|t_{\nu_k}-f(y'_k)|,|t_{\nu_k}-g(y'_k)|\}\geqslant 0$ we have obviously the bound: $d_k\leqslant 1/\nu_k+1/(2\nu_k)+1/(4\nu_k)=7/(4\nu_k)$ (for, at worst, t_{ν_k} is equal to $P_{\nu_k}(y'_{\nu_k})$ or $Q_{\nu_k}(y'_{\nu_k})$). Let us denote by $h_k:=f$ or $h_k:=g$ so that $d_k=|t_{\nu_k}-h_k(y'_k)|$. Then, choose $\tau_k\in(f(y'_k),g(y'_k))$ in such a way that $|\tau_k-h_k(y'_k)|<1/(4\nu_k)$. This implies $|\tau_k-t_{\nu_k}|<7/(4\nu_k)+1/(4\nu_k)=2/\nu_k$. Thus, we put $(x''_k,t'_k):=(y'_k,\tau_k)\in(f,g)$. Clearly, $(x''_k,t'_k)\to(x',t)$, whence $(x',t)\in\overline{(f,g)}$ and so $\limsup C_\nu\subset\overline{(f,g)}$.

Finally, note that $\dim C_\nu=\dim\overline{(f,g)}=\dim(f,g)$.

(iii) Therefore, if we write $E=\bigcup_1^r C_j$ as a finite union of cells C_j, and we approximate each cell $C_j^\nu\to\overline{C_j}$ by compact semialgebraic sets C_j^ν with $\dim C_j^\nu=\dim C_j$, we obtain a sequence of semialgebraic compact sets $A_\nu:=\bigcup_j C_j^\nu$ approximating E and such that $\dim A_\nu\cap U=\dim E\cap U$ for each open set U and ν large enough (since $\dim E=\max_j\dim C_j$ and $\dim C_j\cap U=\dim C_j$, if $C_j\cap U\neq\varnothing$).

(c) The second part of the theorem follows from Theorem 3.9, since the set $(\{0\}\times E)\cup\bigcup_\nu(\{1/\nu\}\times A_\nu)$ satisfies its assumptions. □

Remark 4.2. The compactness assumption is obviously purely technical for the first two assertions to hold. However, the second part of the theorem is no longer true when we drop it. To see this it suffices to consider

$$\left\{(x,y)\in\mathbb{R}^2\;\middle|\;x\in\{1,-1\}\right\}=\lim\left\{(x,y)\in\mathbb{R}^2\;\middle|\;y=\left(x^2-1\right)/\nu\right\}.$$

It is also clear that the theorem is still true if we change the word ‘subanalytic’ to ‘definable’.

In the course of the proof we needed to represent C' as an increasing union of definable compacts. This is explained by the fact that e.g. $f\equiv 0$ may be approximated locally uniformly on $C'=(0,1)$ by $P_\nu(x)=x^\nu$, but $\lim P_\nu\neq\overline{f}$ for the graphs seen in $\mathbb{R}^2$.

In the proof we used the following easy to verify lemma (cf. [9], §29.VI):

Lemma 4.3. *Let $Z=\bigcup K_\nu$, where $K_\nu\subset K_{\nu+1}$. Then $\overline{Z}=\lim K_\nu$ in the sense of Kuratowski.*

Let us note also:

Lemma 4.4. *Let $E\subset\mathbb{R}^m$ be a compact set and $f_\nu, f:E\to\mathbb{R}^n$ be continuous functions. If $f_\nu\to f$ uniformly on E, then $f_\nu(E)\to f(E)$ in the sense of Kuratowski.*

Proof. Take $y\in f(E)$ and pick $x\in E$ so that $f(x)=y$. Since $f_\nu(x)\to f(x)$, we have $f(E)\subset\liminf f_\nu(E)$. On the other hand, if $K\subset\mathbb{R}^n\setminus f(E)$ is a non-void compact set, then $\varepsilon:=\min\{\|y-k\|\colon\ y\in f(E),\ k\in K\}>0$. Since for all but finitely many indices $\|f_\nu-f\|_E<\varepsilon$, then clearly $f_\nu(E)\cap K=\varnothing$. □

Obviously, the converse is not true as can be seen from the example $f_\nu(x)=x^\nu$, $f\equiv 0$ on $E=[0,1]$. It is also clear that the lemma holds true for E a compact topological space and any metric space instead of $\mathbb{R}^n$ (ε is then computed as the distance of two compact sets).

Besides, observe that the same counterexample on $E = (0, 1)$ shows that the result does not hold, if we replace uniform convergence by local uniform convergence in an open set.

Using this lemma we obtain:

Proposition 4.5. *Let $A \subset \mathbb{R}^m$ be a compact semialgebraic set and $E \subset \mathbb{R}^n$ a compact subanalytic set. Suppose that there exist an open neighbourhood $U \supset A$ and an analytic function $f : U \to \mathbb{R}^n$ such that $f(A) = E$. Then f induces a sequence of semialgebraic compact sets $A_\nu \subset V$ converging to E in the sense of Kuratowski.*

Proof. Fix a point $x \in A$ and a compact ball $K_x \subset U$, centred at x, in which f is equal to its Taylor series at x. Let us denote by $T_x^\nu f$ the ν-th partial sum of the Taylor series of f at x. Put $A_x^\nu := T_x^\nu f(A \cap K_x)$. Since $T_x^\nu f$ is a polynomial and $A \cap K_x$ is semialgebraic, each A_x^ν is semialgebraic too, by the Tarski–Seidenberg Theorem. By the previous lemma, $A_x^\nu \to f(A \cap K_x)$ in the sense of Kuratowski.

Since A is compact, there is a finite covering $A \subset \bigcup_1^r K_{x_j}$. Put $A_\nu := \bigcup_1^r A_{x_j}^\nu$, it is semialgebraic and compact. By the Kuratowski continuity of finite unions, we obtain $A_\nu \to \bigcup_1^r f(A \cap K_{x_j}) = f(A) = E$. $\square$

This hints at another possibility of obtaining a semialgebraic approximation of subanalytic closed sets, though this time without additional information about the behaviour of the dimension or the connected components. This is based on the Rectilinearization Theorem of Hironaka that implies — as have already been observed by Pawłucki and Pleśniak in [11] — the following theorem:

Theorem 4.6. *Let $E \subset \mathbb{R}^m$ be a subanalytic closed, nonempty set. Then it can be written as the locally finite union of images of closed cubes $I = \{x \in \mathbb{R}^n \mid |x_j| \leqslant 1,\ j = 1, \ldots, n\}$ (n varies) by analytic mappings.*

Proof. It suffices to adapt the proof of Pawłucki and Pleśniak [11], Corollary 6.2.[28] $\square$

From this we obtain the following local approximation:

Theorem 4.7. *Let $E \subset \mathbb{R}^n$ be a subanalytic closed, nonempty set. Each point $a \in E$ admits a compact subanalytic neighbourhood K such that $E \cap K$ is the limit of a sequence of semialgebraic sets.*

Proof. We may take K to be the closed Euclidean ball. By the previous theorem, for some K there is

$$K \cap E = \bigcup_{j=1}^{r} f_j(I_j),$$

where $I_j := \{x \in \mathbb{R}^{n_j} \mid |x_i| \leqslant 1,\ i = 1, \ldots, n_j\}$ and each f_j is an analytic mapping on $\mathbb{R}^{n_j}$.

By the Weierstrass Approximation Theorem, for each f_j we find a sequence of polynomials $P_{j,\nu}$ approximating f_j uniformly on I_j. Then Lemma 4.4 yields $\bigcup_j P_{j,\nu}(I_j) \to \bigcup_j f_j(I_j)$. By the Tarski–Seidenberg Theorem the approximating sets are semialgebraic. $\square$

5. Converging sequences and connectedness

Proposition 5.1. *Let $E \subset \mathbb{R}_t^k \times \mathbb{R}_x^n$ be closed and suppose that E_t are all connected for $t \in F \setminus \{0\}$, where $F := \pi(E)$ for $\pi(t, x) = t$.*

(1) *If $n = 1$ and $E_0 = \liminf E_t$, then E_0 is connected too.*
(2) *If $E_0 = \lim E_t$ and E_0 is compact, then E_0 is connected, too.*

Proof. Ad (1). Suppose that $E_0 = S_1 \cup S_2$, where S_j are closed and nonempty, and the union is disjoint. Take open intervals $U_j \supset S_j$ whose closures are disjoint. Put (possibly after renumbering) $\varepsilon_1 = \max \overline{U_1} < \min \overline{U_2} =: \varepsilon_2$.

Since $E_0 = \liminf E_t$, there is a neighbourhood V of $0 \in \mathbb{R}^k$ such that for each $t \in V \cap F$, $E_t \cap U_j \neq \varnothing$, $j = 1, 2$. On the other hand, for any point $x \in (\varepsilon_1, \varepsilon_2)$ and any its neighbourhood U_x, by the connectedness of E_t, we have (for $t \neq 0$), $E_t \cap U_x \neq \varnothing$. This implies $(\varepsilon_1, \varepsilon_2) \subset \liminf E_t$ which is a contradiction.

Ad (2). The idea is similar to the first part of the proof of Theorem 3.9. Suppose that $E_0 = S_1 \cup S_2$ where $S_1 \neq S_2$ are two closed, nonempty sets. Choose an $\varepsilon > 0$ so that the closures of the ε-neighbourhoods $U_j^\varepsilon \supset S_j$ are disjoint. Since S_j are compact, then $K_j := \overline{U_j^\varepsilon \setminus U_j^{\varepsilon/2}}$ are compact too.

[28] In [11] the proof is done in the case E is a pure-dimensional subanalytic bounded set, but the general proof being almost identical we omit it here.

Now, since no $x \in K_1 \cup K_2$ belongs to $E_0 = \limsup E_t$, we find a neighbourhood $\Omega \supset K_1 \cup K_2$ and a neighbourhood V of $0 \in \mathbb{R}^k$ such that $(V \times \Omega) \cap E = \varnothing$. Note that $\mathbb{R}^n \setminus K_j = U_j^{\varepsilon/2} \cup (\mathbb{R}^n \setminus \overline{U_j^{\varepsilon}})$ the union being disjoint. Thus, due to the fact that $E_0 = \liminf E_t$, we can find a neighbourhood $W \subset V$ of $0 \in \mathbb{R}^k$ such that for $t \in F \cap W \setminus \{0\}$ we have $E_t \cap U_j^{\varepsilon} \neq \varnothing$ for $j = 1, 2$. But then by connectedness, $E_t \subset U_j^{\varepsilon}$ for $j = 1, 2$ (since $E_t \cap \Omega = \varnothing$ for $t \in W$) which is impossible. □

Note that we cannot replace in the proposition above the limit by the upper limit as is shown by Example 2.14 (the second set from this example). Here (2) is a kind of refinement of Zoretti's result cited in [9], §42.II.

Note also that without a compactness assumption for $n \geqslant 2$ the result is no longer true:

Example 5.2. Consider the set E whose sections for $t \neq 0$ are

$$E_t := \{(x_1, 1/x_1^2) \mid x_1 \in [-1, 1] \setminus [-t, t]\} \cup \{(x_1, 1/t^2) \mid x_1 \in [-t, t]\}$$

and $E_0 := \{(x_1, 1/x_1^2) \mid x_1 \in [-1, 1]\}$.

In the sequel we deal with the following natural question asked by J.-P. Rolin: Suppose that F_ν, F are closed sets in $\mathbb{R}^n$ with $F = \lim F_\nu$ and such that there is a uniform bound $\# cc(F_\nu) \leqslant N$ for all ν. Then, is there $\# cc(F) < +\infty$ and if so, is there any relation with the bound N?

(Since now we are interested especially in the number of connected components of the limit set, we restrict ourselves to converging sequences.)

Without a compacity assumption the answer is negative as we have already seen:

Example 5.3. The connected algebraic sets $F_\nu = \{y = \nu(x^2 - 1)\}$ converge to $\{-1, 1\} \times \mathbb{R}$ in $\mathbb{R}^2$. In the same way we can construct for any k connected graphs converging to F with $\# cc(F) = k$, e.g. for $k = 4$ it suffices to take $F_\nu = \{y = \nu(x^2 - 1)(x^2 - 2)\}$.

It is also possible to obtain $k = +\infty$ from connected F_ν's: let $F_\nu = \{y = \nu \sin x\}$; then $F = \{n\pi \mid n \in \mathbb{Z}\} \times \mathbb{R}$. We may as well ask the sets F_ν be semialgebraic — if a_μ denotes the middle of $[1/(\mu + 1), 1/\mu]$, then F_ν is the set obtained as the union of segments connecting the ends of each interval $[1/(\mu + 1), 1/\mu]$ with the point (a_μ, ν), $\mu = 1, \ldots, \nu$. Then $F = (\{0\} \cup \bigcup \{1/\nu\}) \times \mathbb{R}$.

Actually, the loss of connectedness of the limit is always due to a 'run to infinity'[29] as is shown in the following observation.

Proposition 5.4. *If $F_\nu, F \subset \mathbb{R}^n$ are closed, $F = \lim F_\nu$, all the F_ν are connected, while F is not. Then there exists a sequence of points $x_\nu \in F_\nu$ for which $\|x_\nu\| \to +\infty$.*

Proof. By Proposition 5.1, were F compact, we would have F connected. Therefore, F cannot be bounded and so contains a sequence $y_\nu \in F$ for which $\|y_\nu\| \to +\infty$. Since each point y_ν is the limit of a sequence of points $x_\mu^\nu \in F_\mu$, the lemma follows by a diagonal choice. □

However, the compactness of the limit set plays an important role in preserving connectedness:

Proposition 5.5. *Let $F_\nu, F \subset \mathbb{R}^n$ be closed, $F = \lim F_\nu$ and suppose that there is a uniform bound $\# cc(F_\nu) \leqslant N$. If F is compact, then $\# cc(F) \leqslant N$.*

Proof. By extracting a subsequence and taking a smaller N, if necessary, we may assume that $\# cc(F_\nu) = N$ for all ν. Write F_ν as the union of its connected components $S_1^\nu, \ldots, S_N^\nu$. Since the Kuratowski convergence is metrisable and compact, we may assume, once again extracting subsequences, that each sequence $\{S_j^\nu\}_\nu$ converges to closed set S_j (maybe empty). But then $F_\nu \to \bigcup_1^N S_j$ and by uniqueness of the limit, $F = \bigcup_1^N S_j$. By Proposition 5.1 we know that each S_j is connected. Thus, $\# cc(F) \leqslant N$. □

Example 5.6. It is worth noting that even if the limit is compact, it may not be so for the sets F_ν, as one can see from the example $F_\nu = \{0\} \cup [\nu, +\infty)$ converging to $\{0\}$. Of course, it will be the case, if we assume the sets F_ν are connected, since then for $\mathbb{B}(r) \supset F$ and $K = \partial\mathbb{B}(r)$ there must be $F_\nu \cap K = \varnothing$ and $F_\nu \cap \mathbb{B}(r) \neq \varnothing$, from some index onward.

[29] In the case of subsets of some open set $\Omega \subset \mathbb{R}^n$ it would be a run to the border of Ω computed in the projective line $\mathbb{P}_1$.

Another natural question arising in this context is the following: if all the F_ν are connected, F is not and $F = \lim F_\nu$, can F have a compact connected component?

In general, what would be sufficient to prove is that the following property $(*)$ is satisfied:

For any bounded $S \in cc(F)$ there exists a bounded neighbourhood $U \supset S$ such that

$$\partial U \cap F = \varnothing.$$

Taking $F = \{0\} \cup \bigcup \{1/\nu\}$ and $S = \{0\}$ one sees that it may not be possible to obtain $U \cap [\bigcup cc(F) \setminus S] = \varnothing$.

If $(*)$ is satisfied, then by convergence, $F_\nu \cap U \neq \varnothing$ and $F_\nu \cap \partial U = \varnothing$ for almost all indices, which means that $F_\nu \subset U$, since these are all connected sets. But we have already proved in Proposition 5.1 that this implies that the limit is connected.

If we do not ask U to be bounded, the neighbourhood is easy to find. Indeed, either $S = F$, in which case the statement is trivial ($U := S + \mathbb{B}(\varepsilon)$), or F is disconnected (as we actually assumed), in which case it has a decomposition

$$F = F_1 \cup F_2, \qquad F_1 \cap F_2 = \varnothing, \qquad F_1, F_2 \neq \varnothing$$

and $F_j = U_j \cap F$ for some open sets $U_1, U_2 \subset \mathbb{R}^n$. Clearly, S is entirely contained in one of the sets F_1, F_2, say F_1. Then $U := U_1$ is the neighbourhood sought for.[30]

Example 5.7. An example inspired by Carlo Perrone, whom we warmly thank, shows that unless F is closed, one cannot hope to obtain $(*)$:

Consider in $\mathbb{R}^2$ the set

$$F := \{(0,0)\} \cup \bigcup_{n=1}^{+\infty} \left(\mathbb{R} \times \left\{\frac{1}{n}\right\}\right).$$

Now, $\{(0,0)\}$ is a connected component of F (the proof is given in Example 5.11), but any neighbourhood U of it which satisfies condition $(*)$ must be unbounded (as it has to contain lines $\mathbb{R} \times \{1/\nu\}$ with $\nu \gg 1$). Obviously, F is not closed.

Observe that if F is closed, condition $(*)$ for a bounded $S \in cc(F)$ is equivalent to the following condition $(**)$:

For all $x \in S$ there is a bounded neighbourhood U_x such that

$$\partial U_x \cap (F \setminus S) = \varnothing.$$

Indeed, S is closed in F (as any connected component) and thus in $\mathbb{R}^n$. Since it is bounded, it is compact. Therefore, $(**)$ implies that there is a finite covering $S \subset U_{x_1} \cup \cdots \cup U_{x_r} =: U$ which is a bounded neighbourhood of S satisfying

$$\partial U \cap (F \setminus S) = \varnothing.$$

Obviously $\partial U \cap S = \varnothing$ and so $(*)$ holds. On the other hand, $(*)$ clearly implies $(**)$.

That topological question is certainly worth being stated. There is, however, another approach — suggested by M. Brunella — namely: compactification. This, indeed, yields the following theorem.

Theorem 5.8. *Let $F = \lim F_\nu$ where F_ν are connected sets and assume that F is not connected (F_ν and F are not necessarily compact). Then the family $cc(F)$ cannot contain a compact set.*

Proof. Consider a one point compactification $\widetilde{\mathbb{R}^m} = \mathbb{R}^m \cup \{\infty\}$. Let $\widetilde{F}, \widetilde{F_\nu}$ denote the closures of F, F_ν in $\widetilde{\mathbb{R}^m}$.[31] They all are compact sets and $\widetilde{F_\nu}$ are connected, still. Of course, $\widetilde{F} = \lim \widetilde{F_\nu}$ and, moreover, $\widetilde{F} = F \cup \{\infty\}$ (cf. Proposition 5.4). Therefore, by Zoretti's theorem (see [9], §42.II), $\widetilde{F}$ is connected.

It is easy to see that the connectedness of $\widetilde{F}$ is equivalent (F being closed) to the following[32]:

$$\forall K \subset F \text{ compact}, \quad K \text{ is open in } F \quad \Rightarrow \quad K = \varnothing.$$

Suppose that $S \in cc(F)$ is compact. Let $Z := (S + \mathbb{B}(\varepsilon)) \cap F$. It is an open subset of F containing S and such that $\overline{Z} \subset F$ is compact. By Theorem 6.2.24 in [8] the equivalence relation $\mathcal{R}$ defined in the compact set $\overline{Z}$ by

$$z \mathcal{R} z' \quad \Leftrightarrow \quad z, z' \text{ belong to the same connected component of } \overline{Z}$$

yields a compact space $\overline{Z}/\mathcal{R}$ with a base consisting of open-closed sets. Let $\pi : \overline{Z} \to \overline{Z}/\mathcal{R}$ be the canonical projection.

30 There is $\overline{U \cap F}^F = \overline{U} \cap F$. But $U \cap F = F_1 = F \setminus F_2$ is a closed subset of F, whence $\overline{U} \cap F = U \cap F$. Since $\overline{U} = \partial U \cup U$ is a disjoint union, we obtain $\partial U \cap F = \varnothing$.

31 Note that for any closed set $E \subset \mathbb{R}^n$ there is $\widetilde{E} \setminus \{\infty\} = E$.

32 Cf. any neighbourhood of ∞ is of the form $\mathbb{R}^n \setminus K$ with K compact.

The image $\pi(S) = s$ reduces to a point and $\pi^{-1}(s) = S$ (S is a connected component of $\overline{Z}$). The set $\delta Z := \overline{Z} \setminus Z$ is compact and disjoint with S, whence $\pi(\delta Z)$ is compact too and does not contain s. Therefore, one can find an open-closed set $W \ni s$ such that $W \cap \pi(\delta Z) = \varnothing$. Then $V := \pi^{-1}(W)$ is open-closed, contains S and $V \subset \overline{Z} \setminus \delta Z = Z$.

The set $V \subset F$ obtained in this way is compact (since it is closed in $\overline{Z}$) and open in F (as it is open in Z which is an open subset of F), whence it should be empty, contrary to the fact that it contains S. This contradiction finishes the proof. □

Remark 5.9. Note that here we needed the ambient space to be only a metric, locally compact space with a countable base of the topology (S has then a relatively compact neighbourhood Z in F).

In the course of the proof we have just shown that condition $(*)$ holds for any closed set F. Indeed, $F = V \cup (F \setminus V)$ is a decomposition of F into two open, nonempty, disjoint sets. Since V is bounded, it is clear that one can find a bounded open set U for which $V = U \cap F$. As earlier, it follows that $\partial U \cap F = \varnothing$. In particular this yields another proof of the last theorem.

Remark 5.10. At first sight it seems almost obvious that for a given set $F \subset \mathbb{R}^n$ and a compact $S \in cc(F)$, this same S should be a connected component of $\widetilde{F} := F \cup \{\infty\} \subset \widetilde{\mathbb{R}^n}$. Of course compactness is essential.[33] Nevertheless, any attempt to find a direct proof fails since the obstruction we are confronted with is exactly some property like $(*)$ which we need to show. A compact $S \in cc(F)$ is still a connected component of $\widetilde{F}$ only if we assume that F is closed. Otherwise, Example 5.7 yields a counterexample, as presented hereafter.

Example 5.11. The set F from Example 5.7 is not closed and $\{(0,0)\}$ is a compact connected component of it which is no longer a connected component of $F \cup \{\infty\}$ in the one point compactification of the plane.

- $\{(0,0)\}$ is a connected component of F.

Indeed, let $S \in cc(F)$ be the unique component containing the origin.[34] If there is a point $(x, 1/n) \in S$, then by connectedness, the whole line $\ell_n := \mathbb{R} \times \{1/n\}$ is contained in S. But then any line ℓ_k with $k > n$ is also contained in S. If it were not the case for some ℓ_k, $k > n$, then we would necessarily have $\ell_k \cap S = \varnothing$. Now, $\mathbb{R}^2 \setminus \ell_k$ consists of two disjoint open sets inducing a decomposition of S, contradicting the connectedness of the latter. Therefore, $\ell_k \subset S$ for all $k \geqslant n$. Clearly, this contradicts the connectedness of S.[35] Hence $S = \{(0,0)\}$.

- Consider a one point compactification $\widetilde{\mathbb{R}^2} = \mathbb{R}^2 \cup \{\infty\}$ and the set $\widetilde{F} := F \cup \{\infty\}$. Of course, it is not the closure of F, but by adding the point at infinity we obtain a set which *is connected*. In particular the connected component of the point $(0,0)$ is the set $\widetilde{F}$ itself, i.e. $\{(0,0)\}$ is no longer a connected component!

Indeed, each $\widetilde{\ell_n} := \ell_n \cup \{\infty\}$ is connected and so must be their union — call it L — since ∞ is a common point of all the $\widetilde{\ell_n}$. This implies that $\# cc(\widetilde{F}) \leqslant 2$.[36]

Now, suppose that $\widetilde{F} = D_1 \cup D_2$ is a disjoint union of two open (in $\widetilde{F}$), nonempty sets and, say, $\infty \in D_2$. There is an open set $V \subset \widetilde{\mathbb{R}^2}$ such that $D_2 = V \cap \widetilde{F}$. Since $\infty \in V$, then $V = (\mathbb{R}^2 \setminus K) \cup \{\infty\}$ with a compact set $K \subset \mathbb{R}^2$ due to the definition of the topology of $\widetilde{\mathbb{R}^2}$. On the other hand, $D_1 = U \cap F$ where $U \subset \mathbb{R}^2$ is an open set.

Observe that $L \subset D_2$. Otherwise, $L \cap D_1 \neq \varnothing$ and we would get a decomposition of L ($L \cap D_2 \neq \varnothing$, for $\infty \in L$) contrary to its connectedness.

Therefore, $(0,0)$ cannot belong to D_2 (otherwise $\widetilde{F} = \{(0,0)\} \cup L \subset D_2$ and so $D_1 = \varnothing$). Thus $(0,0) \in D_1$ and so U is a neighbourhood of the origin. However, for any such neighbourhood, there must be $U \cap L \neq \varnothing$, whence $D_1 \cap D_2 \neq \varnothing$. This contradiction ends the proof.

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[33] If we take F to be two different parallel lines in the plane, then $\widetilde{F}$ is a figure-of-eight figure on the sphere and so is connected (while F has two unbounded connected components).

[34] Recall that for a given point $x_0 \in F$ the set $S(x_0) := \bigcup\{S \mid S \subset F\colon\ x_0 \in S,\ S \text{ connected}\}$ is exactly the unique connected component of F containing x_0.

[35] In the same manner: any line lying between two consecutive lines ℓ_n, ℓ_{n+1} induces a decomposition of S.

[36] There is one connected component containing the union L of all the $\widetilde{\ell_n}$ and possibly another one containing the point $(0,0)$.

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